Black holes as self-sustained quantum states and Hawking radiation

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Collapse of a star $\implies$ gravitons contribution becomes dominant
\[ \downarrow \]
superposition of $N$ gravitons ($N \gg$ other species)

- graviton Compton/de Broglie wave-length $\ell = \frac{\hbar}{\mu}$
- ball size $\simeq \ell$
- coupling constant $\alpha = \frac{\ell^2_P}{\ell^2}$
- Newtonian approximation: box potential of constant energy

\[ U_{\text{Newton}}(r) = \alpha \frac{\hbar}{r} \simeq -N \alpha \frac{\hbar}{\ell} \Theta(\ell - r) := U_{\text{box}}(\ell) \]

The graviton ball is self sustained if

\[ E_{\text{escape}} + U_{\text{box}}(\ell) = 0 \iff \frac{1}{N} = \alpha \]
When \( \frac{1}{N} = \alpha \) no graviton can escape \( \Rightarrow \) it’s a BLACK HOLE!

BEC of gravitons confined by a Newtonian potential

- Just one defining characteristic: \( N \)
- Size of BH: \( l = \sqrt{N} \ell_P \)
- Correct area quantisation: \( A \sim N \ell_P^2 \)
- BH mass = sum of \( N \) gravitons: \( M = N \frac{1}{\sqrt{N}} m_P = \sqrt{N} m_P \)
- Depletion of the BEC = Hawking radiation

BHs are macroscopic quantum objects
Quantum horizon formalism (Casadio 2013)

How to apply GR to “macroscopic particles”?

Hoop conjecture $\Rightarrow$ BH is formed if object size $\lesssim 2 \, G \, M \equiv r_H$

Idea: expand a particle wavefunction in its energy eigenstates

$$|\psi_S\rangle = \sum_E C(E)|\psi_E\rangle \quad \hat{H}|\psi_E\rangle = E|\psi_E\rangle$$

$\forall$ energy eigenstate $\exists$ a Schwarzschild radius s.t. $E = \frac{m_P r_H}{2 \ell_P}$

The horizon is not exactly localized, but fuzzy

$$\psi_H(r_H) \sim C \left( \frac{m_P r_H}{2 \ell_P} \right)$$
What is the statistical distribution of the possible horizons $r = r_H$?

$$P_H(r_H)dr_H = 4\pi r_H^2 |\psi_H(r_H)|^2 dr_H$$

What is the probability of finding the particle inside $r_H$?

$$P_S(r < r_H) = 4\pi \int_0^{r_H} dr \ r^2 |\psi_S(r)|^2$$

Probability that particle is a black hole = probability that particle is inside $r_H$ AND that $r_H$ is the horizon FOR ALL possible values of $r_H$

$$P_{BH} = \int_0^\infty P_H(r_H)P_S(r < r_H)dr_H$$
Example: Gaussian wavefunction

\[
\psi_s(r) = \frac{e^{-\frac{r^2}{2\ell^2}}}{\ell^{3/2} \pi^{3/4}} \quad \ell \sim \lambda_m = \frac{\hbar}{m} \text{ (Compton wavelength)}
\]

What is \( \psi_H(r_H) \)? use a Fourier transform!

\[
\psi_s(p) = \frac{e^{-\frac{p^2}{2\Delta^2}}}{\Delta^{3/2} \pi^{3/4}} \quad \Delta = m_P \frac{\ell_P}{\ell} \simeq m
\]

Substitute

\[
E^2 = p^2 + m^2 \quad E = \frac{m_P r_H}{2\ell_P}
\]
⇒ normalised horizon wavefunction

\[ \psi_H(r_H) = \frac{\ell^{3/2} e^{-\frac{\ell^2 r_H^2}{8\ell_P^4}}}{2^{3/2} \pi^{3/4} \ell_P^{3/4}} \]

Using the general expression

\[ P_{BH} = \int_0^\infty P_H(r_H)P_S(r < r_H)dr_H \]

we get the final result

\[ P_{BH}(\ell) = \left[ \arctan \left( 2 \frac{\ell_P^2}{\ell^2} \right) + 2 \frac{\ell^2 (4 - \ell^4/\ell_P^4)}{\ell_P^2 (4 + \ell^4/\ell_P^4)^2} \right] \]
\[ P_{BH}(\ell) = \left[ \arctan \left( 2 \frac{\ell_P^2}{\ell^2} \right) + 2 \frac{\ell^2 (4 - \ell^4 / \ell_P^4)}{\ell_P^2 (4 + \ell^4 / \ell_P^4)^2} \right] \]

Probability that the particle is a BH as a function of \( \frac{\ell}{\ell_P} = \frac{m_P}{m} \).
Quantum uncertainties:

\[ \Delta r^2 \sim \ell^2 \quad \Delta r_H^2 \sim \frac{\ell^4}{\ell^2} \]

Linear combination of uncertainties \( \Rightarrow \) minimum measurable length

\[ \Delta R = \sqrt{\Delta r^2} + \eta \sqrt{\Delta r_H^2} \gtrsim \sqrt{\eta} \ell_P \]

Quantum horizon formalism naturally provides a GUP (no need of modified commutators or new physics!)
Relative uncertainty doesn’t decrease for macroscopic particles

\[ \frac{\Delta r_H}{\langle r_H \rangle} \sim \text{const} \]

One heavy particle \( \neq \) many light particles

\[ \Downarrow \]

Let’s put together \( N \) particles and symmetrise them
Improved model of BEC BH

More realistic model:
BH core + hair

- BEC core $\Rightarrow \begin{cases} 
\text{has minimum allowed momentum } k = k_c \\
\text{makes the discrete spectrum}
\end{cases}$

- Hair $\Rightarrow \begin{cases} 
\text{has continuous spectrum } k > k_c \\
depleted quanta reproduce Hawking radiation? \\
\text{responsible for the fuzziness of the horizon}
\end{cases}$

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Assumption: thermal damping of continuous spectrum

\[ | \psi^{(i)}_S \rangle \simeq \mathcal{N}_\gamma \left( | m \rangle + \gamma \frac{e^{T_H^{-1} \frac{m}{2}}}{\sqrt{T_H}} \int_m^{\infty} dE_i \ e^{-T_H^{-1} E_i} | E_i \rangle \right) \]

- \( \mathcal{N}_\gamma = (1 + \gamma^2)^{-1/2} \)
  - normalisation factor
- \( \gamma \in \mathbb{R} \) and dimensionless:
  - relative probability of finding the particle in the continuous spectrum
- \( T_H = \frac{m_P^2}{4 \pi M} \approx \frac{m_P}{\sqrt{N}} \)
  - since \( M = \sqrt{N} \ m_P \)

\[ C(E_i) \]

Energy of particles in the core

Energy of particles in the hair

\[ E_i \]

\[ m \]
For $N$ particles

**Symmetrised product**

$$|\psi_S\rangle \simeq \frac{1}{N!} \sum_{\{\sigma_i\}}^{N} \left[ \bigotimes_{i=1}^{N} |\psi_S^{(i)}\rangle \right] =$$

$$\frac{1}{N!} \sum_{\{\sigma_i\}}^{N} \left[ \bigotimes_{i=1}^{N} \left( |m\rangle + \gamma \frac{e^{T_H^{-1} m^2}}{\sqrt{T_H}} \int_{m}^{\infty} dE_i \ e^{-T_H^{-1} E_i} |E_i\rangle \right) \right]$$

Expansion in powers of $\gamma \Rightarrow 2$ possible limiting cases

1. $\gamma \ll 1$: most of the energy is in the core
2. $\gamma \geq 1$, $N \gg 1$: most of the energy is in the hair
For $\gamma \ll 1$ (1 boson in the hair is enough)

Energy expectation value (large $N$ and to leading order in $\gamma$)

$$\langle E \rangle \simeq \sqrt{N} \, m_P \left(1 + \frac{\gamma^2}{4 \, e \, N}\right) \quad \Delta E \simeq \frac{\gamma \, m_P}{2 \sqrt{e \, N}}$$

There is a classical limit

$$\frac{\Delta E}{\langle E \rangle} \sim \frac{\gamma}{2 \sqrt{e \, N}} \quad \frac{\Delta r_H}{\langle \hat{r}_H \rangle} \sim \frac{1}{N}$$
Dvali & Gomez model translated to QFT:
QM + Newton $\mapsto$ Massless scalar field + source

Horizon wavefunction formalism tested successfully on other bald and hairy BHs
- Analytical approximation
- Numerical evaluation of large $N$ systems

$\gamma$ parameter might be used to describe the formation of BHs as a phase transition