Quantum field theory on curved spacetime and semiclassical Einstein equations

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Motivations

- **Quantum matter - gravity interplay** should be described within: quantum gravity.

- However at least in some regime it should be possible to analyze it in the semiclassical approximation:
  - Quantum Fields on **fixed** curved spacetimes 
    *(Particle Creation, Hawking Radiation)*
    good for the description of the metric fluctuations.
  - **Backreaction** in a semiclassical fashion
    \[ G_{ab} = 8\pi \langle T_{ab} \rangle \]
    good for the description of "evolution" in cosmological models.
  - It equates **classical** quantities with **probabilistic** ones.

**Question**

Can we solve that equation? Is the solution meaningful?
Plan of the talk

- Quantum Field Theory on Curved Spacetime: the algebraic approach.
- Global existence of solutions of semiclassical gravity in cosmology.
- Metric fluctuations induced (passively) by matter.

This talk is based on

- C. Dappiaggi, K. Fredenhagen, NP PRD 77, 104015 (2008).
- NP, D. Siemssen, CMP online first (2014).
Quantum field theory: The algebraic approach

- Quantization of a (linear)-field theory in flat space is based on the choice of a **Fock space** built over a **vacuum**.

- In curved spacetime:
  - We don’t have a preferred **time**!
  - We don’t have **symmetries** used to construct a vacuum state.
  - Inequivalent representations arise very often.

- Base our theory on the **choice** of observables and on **relations** among them, **without** have recourse to particular state or Hilbert space representation.

- The Quantization is a two Step Process
  - Identify the observables of the theory $\mathcal{A}$,
  - find a state $\omega : \mathcal{A} \rightarrow \mathbb{C}$ to get measurements.
Considered matter

- Massive scalar quantum field conformally coupled to gravity.
  \[-\Box \phi + \frac{1}{6} R \phi + m^2 \phi = P \phi = 0\]

- In a **globally hyperbolic** spacetime $M$ there are unique advanced $\Delta_A$ and retarded $\Delta_R$ fundamental solutions $P \Delta_{A/R} = \delta$.
  \[\Delta = \Delta_A - \Delta_R\]

- **Quantization:** $M \mapsto \mathcal{A}(M)$ the $\ast$-algebra $\mathcal{A}(M)$ generated by linear fields $\varphi(f)$, implementing:
  \[\{ \varphi(f), \varphi(h) \} = i \Delta(f, h)\]
  plus involution and the equation of motion

- It can be understood as a **deformation** of the algebra of functionals over smooth field configurations $\varphi$ **[Brunetti Duetsch Fredenhagen]**
  \[(F \ast G)(\varphi) := \sum_n \frac{1}{n!} \frac{i^n}{2^n} \langle F^{(n)}(\varphi), \Delta \otimes^n G^{(n)}(\varphi) \rangle\]
Regularization

- We need **point wise** products of fields like
  \[ \varphi^2(f), \quad T_{ab}(f) \]

- Their product (used to construct the variance) involves
  \[ \varphi^2(f) \ast \varphi^2(g) = \cdots + "\Delta^2(f, g)" \]

- Solved employing another distribution with equal antisymmetric part
  \[ \frac{i}{2} \Delta \rightarrow \mathcal{H} = \mathcal{H}_s + \frac{i}{2} \Delta \]

- Afterwards we may include Wick powers in the algebra. (by deforming
  the product and then by extending it trivially) ▶ Details.

In flat spacetime \( \mathcal{H} \) can be the vacuum → normal ordering.

It works due to **positivity of the energy density**.
On curved spacetime, we need to choose $\mathcal{H}$ in such a way that:

- It matches **normal ordering** in flat spacetime
- The resulting stress tensor is covariantly **conserved**: $\langle \nabla^a T_{ab} \rangle_\omega = 0$
- It implements **local covariance**

In curved spacetime $\mathcal{H}$ needs to be constructed only out of local quantities. [Hollands, Wald, Brunetti, Fredenhagen]

**Local** version of the **positivity** by means of Microlocal analysis. [Radzikowski]

The freedom is not so big:

$$\mathcal{H} = \frac{U}{\sigma_\epsilon} + V \log \left( \frac{\sigma_\epsilon}{\lambda^2} \right)$$

- It can be tested on **Hadamard states** (states whose n-pt f. have a prescribed UV behavior).
- It implements the **point splitting regularization** (with some freedom).
Point splitting regularization

Hadamard states

\[ \omega_2 = \mathcal{H} + W = \frac{U}{\sigma_\epsilon} + V \log \left( \frac{\sigma_\epsilon}{\lambda^2} \right) + W \]

**Stress-Energy Tensor:** [Moretti 2003]

\[ T_{ab} := \partial_a \varphi \partial_b \varphi - \frac{1}{6} g_{ab} \left( \partial_c \varphi \partial^c \varphi + m^2 \varphi^2 \right) - \frac{1}{6} \nabla (a \partial_b) \varphi^2 \]

\[ + \frac{1}{6} \left( R_{ab} - \frac{R}{6} g_{ab} \right) \varphi^2. \]

**Expectation values:**

\[ \omega(T_{ab}) = \lim_{y \to x} D_{ab} [\omega_2(x, y) - \mathcal{H}(x, y)] \]

There is a **freedom** in the identification of the elements of \( \mathcal{A} \) rep. \( T_{ab} \).
Other choices of $\mathcal{H}$ produce isomorphic theories.

Local fields are determined up to some renormalization freedom.

Assuming some reasonable hypotheses the freedom is finite. \cite{HollandsWald}

\[
\tilde{\varphi}^2 = \varphi^2 + \alpha m^2 + \beta R
\]

\[
\tilde{T} = T + \alpha m^4 + \beta m^2 R + \gamma \Box R.
\]

$\alpha$ expresses a renorm. of the cosmological constant. $\implies (\alpha m^4)$

$\beta$ expresses a renorm. of the Newton constant $\implies (\beta m^2 R)$

$\gamma$ is a pure quantum freedom. $\implies (\gamma R^2)$

Local covariance implies that it suffices to fix ren. freedom. somewhere in order to fix it everywhere. \cite{BrunettiFredenhagenVerch}
Quantum Field Theory on Curved Spacetime

Anomaly and other components of $\langle T \rangle$

\[
\langle \varphi(x)\varphi(y) \rangle_\omega = \omega_2(x, y) - \mathcal{H}(x, y)
\]

- **Covariance** implies that $\mathcal{H}$ is constructed out of local objects only. $\mathcal{H}$ is not a state, $\mathcal{H}$ does not satisfy the equation of motion.

\[
8\pi^2 \langle \varphi P\varphi \rangle_\omega = 6[v_1], \quad 8\pi^2 \langle (\nabla_a \varphi)(P\varphi) \rangle_\omega = 2\nabla_a[v_1]
\]

- In $T$ there are three contributions, the state dependent part, the renormalization freedom and the anomalous term.

\[
\langle T \rangle_\omega = m^2 \langle \varphi^2 \rangle_\omega + \alpha m^4 + \beta m^2 R + \gamma \Box R + T_{\text{anomaly}}.
\]

\[
T_{\text{anomaly}} = \frac{1}{360} \left( C_{ijkl} C^{ijkl} + R_{ij} R^{ij} - \frac{R^2}{3} + \Box R \right) + \frac{m^4}{4} \simeq [v_1]
\]
Semiclassical Einstein equations in cosmology.

- Cosmological spacetimes

\[ (M, g), \quad M = I \times \Sigma . \]

- For flat FLRW spacetime

\[ g = a(\tau)^2 \left[ -d\tau \otimes d\tau + dx^i \otimes dx^i \right], \]

\( a \) is the scale factor and \( \tau \) conformal time. \( H = a'/a^2 \)

- Semiclassical Einstein equation

\[ G_{ab} = \langle T_{ab} \rangle_\omega. \]

- Simpler equation

\[ -R = \langle T \rangle_\omega, \quad \nabla_a T^{ab} = 0, \quad \rho(0) = cH^2(0) \]

- We look for **existence** and **uniqueness** of solutions of that system.
The equation \(-R = 8\pi \langle T \rangle\) is

\[-6 \left( \dot{H} + 2H^2 \right) = -8\pi m^2 \langle \varphi^2 \rangle_\omega - \frac{1}{30\pi} \left( \ddot{H}H^2 + H^4 \right) + \frac{m^4}{4\pi}\]

**Important:** The quantum state enters in the equations via \(\langle \varphi^2 \rangle_\omega\)

**Physical input:** We would like to use “vacuum states” i.e. \(\langle \varphi^2 \rangle_\omega = 0\)

**Impossible**

**Assume** (for the moment) \(T_{state} = 0\)

We have only \(T_{anomaly}\) and \(T_{ren.freedom} = \alpha m^2 + \beta R\)

The differential equation is an ordinary one \(\Rightarrow\) it can be solved!
With some choice of $\alpha$ and $\beta$ $H = 0$ and $H = H_\perp$ are stable solutions.

- $(m = 0)$ a length scale is introduced (proportional to $G$). Two fixed points instead of one.
  
  \cite{Starobinsky, Wald, Vilenkin, Shapiro Sola, Odintsov ...}

- $(m \neq 0)$ $H_\perp$ is a renormalization constant.

- It is not a simple mixture of dust, radiation and dark energy

  $\rho_{\text{anomaly}} = cH^4$

- Changing $\gamma$ the upper fixed point can become unstable \cite{Starobinsky}.
Semiclassical equations

Form of the initial singularity

Question

Where is the singularity \( t_0 \) in the Penrose diagram?

\[ ds^2 = a^2 \left( -d\tau^2 + dx^2 \right). \]

- **Classical solution**
  Radiation dominated:
  \[ \tau = \tau_0 + A(t - t_0)^{1/2} \to \tau_0 \]
  for \( t \to t_0 \)
  Horizon problem.

- **Quantum Corrections**
  \[ \rho = 1/\rho^2(t) \]
  \[ \tau = \tau_0 + \log(t - t_0) \to -\infty \]
  for \( t \to t_0 \)
  Singularity is light like.

Power law inflation with Null Big Bang (NBB)
Semiclassical equations

Where we need to put initial conditions

Question

What about the exact solutions? (Taking into account $\langle \phi^2 \rangle_\omega$)

- Previous numerical results obtained by Anderson.

- We need a rule to prescribe a state on every FRW spacetime and then use the semiclassical equation to select the correct one.

$$-R = \langle T \rangle_\omega$$

- Impose initial conditions at finite time, say $\tau = \tau_0$ (where the spacetime is regular).
How to construct a state on a cosmological spacetime: brief history

- **Parker**: Adiabatic states.
- **Lüders Roberts**: On firm mathematical ground.
- **Junker Schrohe**: Studied the regularity of those states (not Hadamard).
- **Olbermann**: Smearing in times. Construct combination of modes producing Hadamard states.
- **Dappiaggi Moretti np**: Regular states out of asymptotic properties.
- **Sorkin Johnson**: Spectral properties of causal propagators.
- **Brum Fredenhagen**: Its smeared version.
At \( \tau = \tau_0 \) we fix the state to be as close as possible to the vacuum.

The pure, homogeneous and isotropic Gaussian state

\[
\omega_2(x, y) := \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\chi_k(x_0)}{a(x_0)} \frac{\chi_k(y_0)}{a(y_0)} e^{ik \cdot (x-y)} dk ,
\]

\[
\chi''_k(\tau) + (k^2 + m^2 a(\tau)^2) \chi_k(\tau) = 0 ,
\]

\[
\frac{d}{d\tau} \chi_k - \frac{d}{d\tau} \bar{\chi}_k \chi_k = i .
\]

In order to be close to the vacuum at \( \tau = \tau_0 \) we fix

\[
\chi_k(\tau_0) = \frac{1}{\sqrt{2k_0}} , \quad \chi'_k(\tau_0) = -i \sqrt{\frac{k_0}{2}} , \quad k_0 = \sqrt{k^2 + m^2 a_0^2}
\]

It is an adiabatic state of order 0 at \( \tau = 0 \). (Although it is not Hadamard it is sufficiently regular to construct \( T \))
After fixing the renormalization freedom ($\alpha$, $\beta$ and $\gamma = -1/360$) we obtain a Volterra-functional equation.

For simplicity, in this talk, we shall assume that the anomalous part vanishes and that $\Lambda = 0$. (Full details in [np, Siemssen 2014]

\[ a'(\tau) = a'_0 + m^2 \int_{\tau_0}^{\tau} \langle \varphi^2 \rangle_\omega a^3 \, d\eta \]

\[ \langle \varphi^2 \rangle_\omega = \frac{1}{2\pi^2 a^2} \int_0^\infty \left[ \chi_0 \chi_k - \frac{1}{\sqrt{k^2 + m^2 a^2}} \right] k^2 \, dk \]

\[ \chi''_k(\tau) + (m^2 a(\tau)^2 + k^2)\chi_k(\tau) = 0, \]
Local existence

\( \langle \varphi^2 \rangle_\omega \) can be controlled by \( a \) and \( a' \).

**Proposition**

*Fix* \( a_0, a'_0 \) *and the state at* \( \tau_0 \). *An unique solution* \( a_1 \) *exists in* \( I = [\tau_0, \tau_1) \) *for some* \( \tau_1 > \tau_0 \).*

Proved applying Banach fixed point theorem to the Volterra like equation.

The estimates permit to construct a contraction map.
Proposition

Let $a_I$ in $l = [\tau_0, \tau_1)$ be a solution then, if $a'(\tau_1)$ does not diverge and $a(\tau_1) > 0$ the solution can be extended further in a unique way to $a_J$ with $I \subset J$.

We can order all the solutions $a_I$. $a_I \leq a_J$ is $I \subset J \implies$ a maximal solution exists

Proposition

The maximal solution is unique because of the unique extension.

Summarizing: fixing the initial condition, either the solution exists till infinity or a singularity is encountered. ($a = 0$, $a' = \infty$)
Other initial values

- Changing the initial values \( a_0, a'_0 \) corresponds to change the state.

\[ \chi_{k,1} = A\chi_k + B\overline{\chi}_k \]

- The propositions about existence and uniqueness of the solution, can be generalized to this case provided the state constructed out of \( \chi_{k,1} \) is sufficiently close to \( \omega \) (\( B \) suff. reg.).

- The obtained solution is unfortunately only \( C^2 \). *(work in progress)*

- The employed estimates for \( \langle \varphi^2 \rangle_{\omega} \) do not permit to control the global behavior from the initial conditions. *(work in progress)*
Let's look again at

\[ G_{ab} = \langle T_{ab} \rangle_\omega \]

The right hand side is the mean exp. value of a stochastic variable.

It is an effective equation, valid when "fluctuations" are negligible.

\[ \lim_{x \to y} \langle T_{ab}(x) T_{ab}(y) \rangle_\omega \] diverges

If the variance of \( \langle T_{ab} \rangle_\omega \) is not negligible, like for the Brownian motion \( \implies \) the equation could make sense as a stochastic one.
**Stochastic approach**

- Einstein-Langevin equation [Verdaguer], namely an equation among probability distributions

\[ G_{ab}(x) = T_{ab}(x) \]

- Study the **passive** influence of matter fluctuations on curvature fluctuations.
- It is not easy to compute the probability distribution for \( T_{ab} \) in \( \omega \).

\[ \delta G_{ab} = G_{ab} - \langle G_{ab} \rangle, \quad \delta T_{ab} = T_{ab} - \langle T_{ab} \rangle_\omega, \quad \delta G_{ab} = \delta T_{ab} \]

- The correlations of \( T_{ab}(x) \) are **more complicated** than in Wiener processes or Brownian motions.
- We can equate their moments:

\[ \langle G_{ab}(x) \rangle = \langle T_{ab}(x) \rangle_\omega \]

\[ \langle \delta G_{ab}(x_1)\delta G_{cd}(x_2) \rangle = \langle \delta T_{ab}(x_1)\delta T_{cd}(x_2) \rangle_\omega \]

\[ \ldots \]

\[ \langle \delta G^n(x_1, \ldots, x_n) \rangle = \langle \delta T^n(x_1, \ldots, x_n) \rangle_\omega \]
Question

What is the impact of curvature fluctuations induced by matter on the spacetime?

- We shall analyze it in a toy model: Metric perturbations induced by matter through the semiclassical Einstein equation.

- [Verdaguer] obtained a scale free spectrum of the metric fluctuations (Bardeen potentials) considering a “linearized version” of $\omega_2$ as a source.
Perturbations around a de Sitter spacetime

- We start with a de Sitter spacetime
  \[ \bar{g} = \frac{1}{(H\tau)^2} \left( -d\tau^2 + d\vec{x}^2 \right) \]

- Let's add **Newtonian perturbations** \( \bar{g} \rightarrow g = \bar{g} + \epsilon \tilde{g} : \)
  \[ g = \frac{1}{(H\tau)^2} \left( -(1 + 2\Psi) d\tau^2 + (1 - 2\Psi) d\vec{x}^2 \right) \]

- Linear perturbation of **scalar curvature**
  \[ \delta G = \bar{g}^{ab} (G_{ab} - \langle G_{ab} \rangle) = 6(H\tau)^4 \left( \frac{\partial^2}{\partial \tau^2} - \frac{1}{3} \nabla^2 \right) \frac{\Psi}{(H\tau)^2} \]

- Inverting (with retarded propagator) we get [Ford Woodard]

\[ \langle \Psi(x_1)\Psi(x_2) \rangle = m^4 \left( \begin{array}{c}
\bullet\bullet\bullet\rightarrow\bullet \quad + \quad \bullet\bullet\bullet\rightarrow\bullet \\
\bullet\leftarrow\bullet\rightarrow\bullet \quad + \quad \bullet\leftarrow\bullet\rightarrow\bullet
\end{array} \right) \]
Fluctuations

Power spectrum of $\Psi$

Is obtained computing the spatial Fourier transform of $\langle \Psi(x_1)\Psi(x_2) \rangle$

where the state is

$$\omega_2(x_1, x_2) = \frac{U}{\sigma_\epsilon} + \text{less sing.} = H^2 \tau_1 \tau_2 \omega_M(x_1, x_2) + \text{less sing.}$$

and the square of the two-point function

$$\widehat{\omega}_2^2(\tau_1, \tau_2, \vec{k}) = \frac{1}{16\pi^2} \int_k \infty e^{-ip(\tau_1-\tau_2)} dp$$

We can consider its contribution to the power spectrum

$$\langle \hat{\Psi}(\tau, k)\hat{\Psi}(\tau, k) \rangle \approx \frac{1}{k^3} P_0(k\tau)$$

Non gaussianities arise naturally

$$\langle \hat{\Psi}(\tau, k_1)\hat{\Psi}(\tau, k_2)\hat{\Psi}(\tau, k_3) \rangle \approx \frac{1}{k_1^2 k_2^2 k_3^2} B_0(k_1\tau, k_2\tau, k_3\tau)$$
The rescaled power spectrum $\mathcal{P}_0(k\tau)$
The state of matter fields induce a state for the fluctuations of the newtonian potential $\Psi$.

The state of matter was already chosen in order for the background to solve the semiclassical Einstein equation.

This provides a link with the standard picture.

It could be used to select the form of the quantum state for linearized perturbations.

[Dappiaggi Melati] shown that it works also in asymptotically de Sitter spacetime.
Summary

- Quantum field theory on curved spacetime.
- Semiclassical Backreaction can be analyzed as a well posed problem in cosmology.
- Beyond mean field approximation it can be understood as a stochastic equation.

Thanks a lot for your attention!
Metric perturbation and fluctuation

- Usually linearized gauge invariant perturbations of the inflaton-gravity system are quantized. [Mukhanov, Feldman, Brendenberger 1992]

- We cannot use this approach in semiclassical gravity: We have already quantized the system and we have put ourself in a regime were the quantum nature of gravity is not necessary.

- There is however a way out, namely fluctuations of matter are naturally present in the stress tensor.

- We could evaluate their passive influence on the right hand side $\langle G \rangle$. 
Deformation

- **Deformation:** if we use $-2i\mathcal{H}$ in the place of $\Delta$ to construct a star product $\star_{\mathcal{H}}$.
- It is realized by
  \[
  \alpha_{\mathcal{H}} : (A, \star) \to (A, \star_{\mathcal{H}})
  \]
  such that
  \[
  F \star_{\mathcal{H}} G = \alpha_{\mathcal{H}}(\alpha^{-1}_{\mathcal{H}}(F) \star \alpha^{-1}_{\mathcal{H}}(G))
  \]
- $\alpha_{\mathcal{H}}$ is an isomorphism.
- We can push forward states on $(A, \star)$ to $(A, \star_{\mathcal{H}})$. For $\omega_2$
  \[
  : \omega_2 : := \alpha^{-1}_{\mathcal{H}}(\omega_2) = (\omega_2 - \mathcal{H})_s
  \]
- The scheme incorporates the point splitting regularization
Extended algebra

We can now safely extend \((\mathcal{A}, \star_\mathcal{H})\) to local functionals.

- \(\mathcal{A}_e\): a finite number of non vanishing func. der.
- \(F \in \mathcal{A}_e\), \(F^{(n)}(0)\): compact supp., symmetric, \(\text{WF}(F^{(n)}(0)) \cap \overline{V_+} \cup \overline{V_-} = \emptyset\).

\[
(F \star_\mathcal{H} G)(\varphi) := \sum_n \frac{1}{n!} \langle F^{(n)}(\varphi), \mathcal{H} \otimes^n G^{(n)}(\varphi) \rangle
\]

Hörmander criterion for multiplication of distributions holds.

- \((\mathcal{A}_e, \star_\mathcal{H})\) is the extended \(*\)-algebra of fields.
  
  [Brunetti Dütsch Fredenhagen Hollands Wald]

- They satisfy the axiom of a **Locally covariant theory**
Regularization freedom and ambiguity

- The product in $A_e$ is constructed out of $\mathcal{H}$, there are **ambiguities** (renormalization freedom).
- The algebras for different $\mathcal{H}$ are isomorphic.
- But local fields are not invariant

$$\tilde{\varphi}^k(x) = \varphi^k(x) + \sum_{i=1}^{k-2} C_i(x) \varphi^i(x)$$

- $C_i(x)$: real polynomials of the metric. [Hollands Wald]
- With the correct scaling under rigid dilation

$$C_i \rightarrow \lambda^i C_i$$
Analysis of $\varphi^2$

How $\langle \varphi^2 \rangle_{\omega_1,0}$ depends on $H$?

$$\langle \varphi^2(x) \rangle_{\omega_1,0} = \lim_{y \to x} [\omega_{1,0}(x, y) - \mathcal{H}(x, y)] + \alpha R + \beta m^2$$

Remind: Prescription for fixing the renormalization freedom:

- Minkowskian spacetime on Minkowskian vacuum, fixes $\beta$.
- $\alpha$ changes the value of $H_c$ or $X_c \implies H_c$ is a ren. constant.

We regularize on Minkowski spacetime the problem $-\Box_M + (ma)^2$

$$\lim_{y \to x} \mathcal{H}(y, x) - \frac{1}{a(\tau_1)a(\tau_2)} \mathcal{H}_M(y, x) = \frac{m^2}{8\pi^2} \log a + \alpha' R ,$$

Point splitting at fixed time, then it is enough to subtract

$$\mathcal{H}_M^0(y, x) := \frac{1}{(4\pi)^2} \left( \frac{2}{\sigma_\epsilon} + m^2 a(\tau_x)^2 \log \left( \frac{\sigma_\epsilon}{\lambda^2} \right) \right)$$
Comparison with the first order adiabatic approximation

\[ \mathcal{H}_M^0(y, x) - \frac{1}{(2\pi)^3} \int \frac{e^{ik(y-x)}}{2\sqrt{k^2 + m^2 a(\tau)^2}} \ d^3k \]

is a continuous function

\[ \langle \varphi^2 \rangle_{\omega_{1,0}} := \frac{1}{2\pi^2 a^2} \int_0^\infty k^2 dk \left[ \overline{\chi}_k \chi_k - \Theta(k - ma) \left( \frac{1}{2k} - \frac{m^2 a^2}{4k^3} \right) \right] - \frac{m^2}{8\pi^2} + \alpha R, \]
Consider

$$\Phi(X) := 2\pi^2 a^2 \langle \varphi^2 \rangle_{\omega_{1,0}}$$

**Proposition**

On $B_c$

$$|\Phi(X)(t)| \leq C_1(a_0, t_0, c)t ,$$

$$|D\Phi(X, \delta)| \leq C_2(a_0, t_0, c)t_0 \|\delta\| ,$$

where at fixed $\lambda$, $C_1(\lambda t_0, t_0, c)$ and $C_2(\lambda t_0, t_0, c)$ are bounded for $t_0 \to 0$. 
Construction of the $\chi$

$$\chi''_k + (k^2 + m^2 a_0^2) \chi + \lambda^2 m^2 (a^2 - a_0^2) \chi = 0$$

Perturbative const. over the massless solution $\chi^0_k(\tau) = \frac{e^{-ik_0(\tau-\tau_0)}}{\sqrt{2k_0}}$

$$\chi_k = \sum_{n=0}^{\infty} \chi^n_k$$

$$\chi^n_k(\tau) = -\int_{\tau_0}^{\tau} \sin(k_0(\tau - \tau')) \left( \frac{a(\tau')^2 - a_0^2}{k_0} \right) m^2 \chi^{n-1}_k(\tau') \, d\tau' ,$$

**Proposition**

*The series converges absolutely on $[\tau_0, \tau]$.*

Every $\chi^n_k$ is $O(\lambda^{2n})$
Point splitting regularization is equivalent to

\[ \langle \varphi^2 \rangle_\omega = C \int_0^\infty k^2 \, dk \left[ \chi_k \chi_k - \frac{1}{\sqrt{k^2 + m^2 a^2}} \right], \]

- Small \( k \) do not create problems
- Expand \( \chi_k \) in powers of \( m^2 \)
- The large powers decay in \( k \) sufficiently rapidly
- \( O(\lambda^0) \) vanishes
- \( O(\lambda^2) \) and \( O(\lambda^4) \) and the reminder can be analyzed directly
Construction of the $\chi$

\[ \chi_k'' + (k^2 + m^2 a_0^2) \chi + \lambda^2 m^2 (a^2 - a_0^2) \chi = 0 \]

Perturbative const. over the massless solution $\chi_0^0(\tau) = \frac{e^{-ik_0(\tau-\tau_0)}}{\sqrt{2k_0}}$

\[ \chi_k = \sum_{n=0}^{\infty} \chi^n_k \]

\[ \chi^n_k(\tau) = -\int_{\tau_0}^{\tau} \frac{\sin(k_0(\tau-\tau'))}{k_0} \left( a(\tau')^2 - a_0^2 \right) m^2 \chi^{n-1}_k(\tau') \, d\tau' , \]

**Proposition**

*The series converges absolutely on $[\tau_0, \tau]$.*

Every $\chi^n_k$ is $O(\lambda^{2n})$
Proposition

We can control $\langle \varphi^2 \rangle_\omega$ as well as its (first-)functional derivative w.r.t. $a'$ on $C^0(\tau_0, \tau)$.

$\|\langle \varphi^2 \rangle_\omega \|_\infty \leq c(\|a'\|_\infty, \tau - \tau_0), \quad \|D\langle \varphi^2 \rangle_\omega \|_\infty \leq c(\|a'\|_\infty, \tau - \tau_0)\|\delta a'\|_\infty$

\[
\langle \varphi^2 \rangle_\omega = \frac{1}{2\pi^2 a^2} \int_0^\infty \left[ \frac{1}{\sqrt{k^2 + m^2 a^2}} \right] k^2 dk
\]

- Small $k$ do not create problems
- Expand $\chi_k$ in powers of $m^2$
- The large powers decay in $k$ sufficiently rapidly
- $O(\lambda^0)$ vanishes
- $O(\lambda^2)$ and $O(\lambda^4)$ and the reminder can be analyzed directly

→ Back.
Proposition

If $t_0$ is sufficiently small, the map $\mathcal{T}$ is a contraction in $B_c$.

Proof

$$\| \mathcal{T}(y) - \mathcal{T}(x) \| \leq C \| y - x \| \quad C < 1$$

$$x_\lambda = x + \lambda \delta \text{ with } \delta = (y - x)$$

$$\left| \frac{d\mathcal{T}}{dt} (y) - \frac{d\mathcal{T}}{dt} (x) \right| \leq \left| \int_0^1 \frac{d}{d\lambda} \frac{d\mathcal{T}}{dt} (x_\lambda) \, d\lambda \right| \leq \sup_{\lambda \in [0,1]} \left| D \frac{d\mathcal{T}}{dt} (x_\lambda, \delta) \right|$$

Theorem

$X_0 := H_0^{-1}$ the massless solution then the sequence

$$X_n := \mathcal{T} (X_{n-1})$$

converges in $B_c$ to the solution $X = H^{-1}$ of the semiclassical Einstein equation.
Some comments

- The found solution can be shown to be $C^1$ only

- **But** there is always a smooth spacetime as close as you want to the solution

- The existence does not depend on the state, in the sense that other $\omega_{A,B}$ with $B$ rapidly decreasing produce a solution.

- All these solutions show a phase typical of power law inflation which is then state independent.

- The found solution does not depend on $a_0$

- It can be be fixed a posteriori by $G_{00} = 8\pi \langle T_{00} \rangle$
Analysis of the fluctuations

The solution is meaningful provided the variance of $T_{\mu}^\mu$ is small

- The anomaly is a $C$–number
- The variance of $\langle \phi^2 \rangle$

$$\Delta_\omega(\phi^2) := \omega(\phi^2 \star_H \phi^2) - \omega(\phi^2)\omega(\phi^2)$$

**diverges:** it is proportional to $\omega_2 \cdot \omega_2(x, x)$

When smeared the situation is better, consider the family centered in $x_\tau$

$$f_{n_1, n_2}(\tau', x) = \frac{n_1}{n_2^3} f \left( \frac{n_1(\tau' - \tau) + \tau, x}{n_2} \right)$$

where

$$f(x_\tau) = 1 , \quad \int_M f \, d\mu(g) = 1 , \quad f \geq 0$$
We study the limit

\[ \lim_{n_1 \to \infty} \lim_{n_2 \to \infty} [R(f_{n_1,n_2}) + 8\pi \langle T \rangle \omega(f_{n_1,n_2})] = R(x_\tau) + 8\pi \langle T \rangle \omega(x_\tau) \]

**Theorem**

*We have*

\[ \lim_{n_2 \to \infty} \Delta_{\omega_1,0}(\varphi^2(f_{n_1,n_2})) = 0 . \]

- In a **weaker** sense, the solution we have found is meaningful also when \( H \) is very large.
CMB Temperature fluctuations

- CMB anisotropies observed by the Planck space telescope.
- Produced at the time of matter/radiation decoupling.
- Usually explained by inflation.
Variance

CMB Temperature fluctuations

\[ \Theta(\tau, \vec{x}, \vec{e}) = \frac{\delta T(\tau, \vec{x}, \vec{e})}{T(\tau)} = \sum_{\ell, m} \Theta_{\ell m}(\tau, \vec{x}) Y_{\ell m}(\vec{e}) \]

\( \Theta_{\ell m}(\tau_0, \vec{x}_0) \) are statistically homogeneous random variables with correlations

\[ \langle \Theta_{\ell m}(\tau_0, \vec{x}_0) \Theta_{\ell' m'}(\tau_0, \vec{x}_0)^* \rangle = \delta_{\ell \ell'} \delta_{m m'} C_{\ell} \]

where, in terms of Newtonian perturbations \( \Psi \),

\[ C_{\ell} = 4\pi \int_0^{\infty} T_\ell(k)^2 \langle \hat{\Psi}(\tau_1, k) \hat{\Psi}(\tau_1, k) \rangle k^2 \, dk \]

In order to be coherent with observations, for small \( k \) it should be

\[ \langle \hat{\Psi}(\tau_1, k) \hat{\Psi}(\tau_1, k) \rangle \approx C \left( \frac{k_0}{k} \right)^{3-\epsilon} \]

we shall compare this with results obtained in an inflationary model.